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Dr. A. C. D. CROMMELIN, PRESIDENT, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

William Hunter McCrea, B.A., Trinity College, Cambridge ;
Edgar Girard Croker Poole, M.A., D.Phil., Fellow and Tutor of New College, Oxford ;
William Arthur Doubleday Smith, B.Sc., 35 Griffin Road, London, S.E. 18 ;
Arthur Thomas Wadley, M.Sc., 11 Tyrwhitt Road, London, S.E. 4 ; and
The Rev. Eber Edward Wright, 115 Carlton Hill, Nottingham.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Professor W. E. Anderson, Miami University, Florida, U.S.A. (proposed by J. J. Nassau) ;
Gorakh Prasad, D.Sc., Reader in Mathematics, University of Allahabad, India (proposed by R. A. Sampson) ;
Ralph Leslie Robinson, B.A., M.R.C.S., L.R.C.P., 30 The Parks, Minehead, Somerset (proposed by A. M. Newbegin) ; and
T. T. Whitehead, 2 Valley Terrace, Simonstone, near Burnley, Lancs (proposed by S. Chapman).

Seventy-seven presents were announced as having been received since the last meeting, including, amongst others :—

T. W. Backhouse, Star Map—14 Charts on the Gnomonic Projection. Engraved by the late George Pearson (presented by Mrs. Pearson).

T. C. Chamberlin, The Two Solar Families.

Helsingfors Observatory, Astrographic Catalogue, tome 1, fasc. 1 ; tome 6.

Rome, Specola Vaticana, Astrographic Chart, zones + 55°, + 56° (10 charts).

Professor Ejnar Hertzsprung delivered the George Darwin Lecture, taking as his subject "The Pleiades."

The Formation of Absorption Lines. By A. S. Eddington, F.R.S.

1. The theoretical calculation of intensities and contours of absorption lines in stellar spectra consists of two parts—

(*a*) The determination of the coefficients of absorption and emission of the material, according to the theories of atomic physics.

(*b*) The study of the outward flow of radiation through an atmosphere composed of such material.

The main recent advance in the subject falls under the first head. Unsöld has introduced into astronomical investigations a formula for the atomic absorption coefficient which has generally superseded earlier estimates. In particular it gives a law of decrease of the absorption coefficient on either side of the centre of the line and so directs attention to the problem of line-contours. This change, however, does not affect the investigation (*b*), and the formulæ already developed are not superseded.

In the many interesting applications of Unsöld's formula there has been a tendency to revert to cruder treatment of the problem (*b*). I think that the solution of this problem in *Internal Constitution of the Stars*,* §§ 234–238, is preferable to those which have been published since. The formulæ are not more complicated than those obtained by Milne with the Schuster-Schwarzschild approximation, and there is no advantage in using the latter approximation when a closer approximation is as handy. Owing to the advances under (*a*) and the extended range of practicable problems that now attract interest, it seems desirable to set forth this theory again with the elaborations and illustrations that now are called for.

§ 3 contains the general equations, and § 4 the solution for a "standard case." We then consider successively the more important deviations from the standard case. Only one type of deviation is considered at a time, the other conditions remaining as in the standard

* Hereinafter referred to as *I.C.S.*

case. Special reference may be made to the diagrams showing the effect of finite layers of absorbing material in § 8, and to the solution for a case of varying opacity in § 10.

From § 13 onwards we show that, at any rate so far as the main part of the absorption line is concerned, the problem of determining line-intensity remains unsolved except in unusually simple spectra where there is no interlocking. Warning is given against forming any "expectation" of the relative intensities of dark lines, *e.g.* in the Balmer series. The absorption coefficient has very little direct effect in determining the strength of a line; its indirect effect is doubtless important but incalculable.

In § 15 we state the implicit physical assumptions involved in Unsöld's theory if (as seems to be generally assumed) it permits the intensities of the wings of spectral lines to be calculated without regard to the interlocking which affects the intensities of the more central parts of the lines.

2. Before restating the theory, I make the following remarks on the investigation as it stands in my book. The formulæ there given for the intensity (the "blackening") are applicable to the separate parts of a line as well as to the line as a whole. This was not stated explicitly because it was unknown (and deemed unlikely) at the time; it results from the adoption of Unsöld's theory of the absorbing process. The crucial question is whether light absorbed in one part of a line is re-emitted in precisely the same part of the line. If so, the blackening in this frequency is independent of what is happening in neighbouring frequencies. The alternative is that the re-emission has a probability distribution, and is correlated to, but not determined by, the absorbed frequency. For example, if the process is regarded as one of transition between two energy levels, which are not sharp but are composed of narrow bands of energy, the atom is not likely to return to the precise spot in the lower level from which it started, and the re-emission will not be the exact reverse of the absorption. In that case the line can only be studied as a whole. Modern attempts to interpret the contours of absorption lines assume (rightly or wrongly *) that there is no such redistribution of frequencies.

Many writers now use the term "scattering" instead of "absorption" for the atomic process which forms spectral lines. Different views are held as to whether there is a physical distinction between line-absorption with re-emission and line-scattering. The difference (if any) does not affect the calculation of intensity of the spectral lines, and, as formerly, I include all scattering coefficients in the absorption coefficient.

The quantitative calculation of the strength of a line is capable of almost indefinite complication according to the conditions assumed. What is first required is not an elaborate formula providing for all contingencies, but a standard solution for the formation of a line under the

* As I am concerned only with problem (*b*), I take the conclusions with regard to (*a*) which are adopted by recent investigators. Some of them may be rather precarious. If the above assumption is untrue, the usual treatment of line-contour is entirely unsound.

most simplified conditions which do not involve a self-inconsistency. This standard solution, given in *I.C.S.*, § 234, is obtained with slightly modified notation in § 4. From it we proceed to consider in turn the more important complications likely to occur in actual stars, and compare the results with the standard.

The standard chosen corresponds as nearly as possible to thermodynamic equilibrium. It is especially desirable to understand the growth of an absorption line in this case. In many qualitative arguments about the strength of lines, authors, whilst professedly neglecting deviations from thermodynamic equilibrium, have actually had in mind as their guide the experiment of passing sunlight through a sodium flame—where the conditions are as remote from equilibrium as can well be imagined. There is practically no resemblance between the growth of the line in the sodium flame and in a star ; the diagrams in § 8 are intended to make the unlikeness more apprehensible.

3. Let k be the coefficient of continuous absorption applicable both inside and immediately outside the absorption line ;
l (a function of ν) the coefficient of line-absorption ;
$J_\nu'(\theta)d\nu$ the flow of radiation of frequency ν to $\nu + d\nu$ within the line in a direction making an angle θ with the outward direction ;
$J(\theta)d\nu$ the corresponding flow just outside the line ;
$d\omega$ the element of solid angle.

We write

$$J = \frac{1}{4\pi}\int J(\theta)d\omega, \quad H = \frac{1}{4\pi}\int J(\theta)\cos\theta d\omega, \quad K = \frac{1}{4\pi}\int J(\theta)\cos^2\theta d\omega \qquad (1)$$

with similar expressions for $J_\nu'(\theta)$. We shall make use of the approximation

$$K = \tfrac{1}{3}J \qquad (2)$$

that is to say, the mean value of $\cos^2\theta$ for the radiation is taken to be $\frac{1}{3}$ instead of its exactly weighted value. It is well known that this is much superior to the Schuster-Schwarzschild approximation. We write

$$\tau = -\int k\rho dr \qquad (3)$$

so that τ is the optical depth for light just outside the absorption line.

In a length ds the radiation $J_\nu'(\theta)$ suffers the continuous absorption $kJ_\nu'(\theta)\rho ds$, and in addition the line-absorption $lJ_\nu'(\theta)\rho ds$. Most of the latter is re-emitted in the same frequency, but a certain fraction ϵ is lost by superelastic collisions. Accordingly the re-emission is $(1 - \epsilon)lJ_\nu'\rho ds$. (Note that J_ν', not $J_\nu'(\theta)$, appears, because the emission is equal in all directions.)

Additional line-emission is given by atoms excited by inelastic collisions. This depends only on the temperature of the material and not on the radiation traversing it (so long as only a small fraction of the atoms are excited at any one moment). If J_T is the equilibrium value

of J_ν' (or J) for the temperature T of the material, the additional emission is

$$\epsilon l J_T \rho ds$$

because in thermodynamical equilibrium the loss by superelastic collisions $\epsilon l J_\nu' \rho ds$ reduces to $\epsilon l J_T \rho ds$, and the gain by the reverse process of inelastic collisions exactly balances it.

In addition to the line-emission there is the ordinary continuous emission $k J_T \rho ds$ for material at temperature T.

Combining the various gains and losses enumerated above, we have

$$\cos\theta \frac{dJ_\nu'(\theta)}{\rho dr} = -(k+l)J_\nu'(\theta) + (1-\epsilon)lJ_\nu' + \epsilon l J_T + k J_T. \quad (4)$$

since $ds = dr \sec\theta$. Multiplying (4) by $d\omega/4\pi$, and also by $d\omega \cos\theta/4\pi$ and integrating, we have by (1)

$$\frac{dH_\nu'}{\rho dr} = -(k+\epsilon l)(J_\nu' - J_T)$$

$$\frac{dK_\nu'}{\rho dr} = -(k+l)H_\nu'.$$

We set

$$\eta = l/k \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Then by (2) and (3) these equations become

$$\frac{dH_\nu'}{d\tau} = (1+\epsilon\eta)(J_\nu' - J_T) \quad . \quad . \quad . \quad (6)$$

$$\frac{dJ_\nu'}{d\tau} = 3(1+\eta)H_\nu' \quad . \quad . \quad . \quad . \quad (7)$$

so that if η is constant

$$\frac{d^2J_\nu'}{d\tau^2} = q^2(J_\nu' - J_T) \quad . \quad . \quad . \quad . \quad (8)$$

where

$$q^2 = 3(1+\eta)(1+\epsilon\eta) \quad . \quad . \quad . \quad (9)$$

4. For our first problem we take the case $J = J_T$, that is to say, the radiation just outside the line has equilibrium density. I have pointed out (*I.C.S.*, p. 338) that actually in a stellar atmosphere J is greater than J_T for one end of the spectrum and less for the other end, so that the case here considered is quite typical. For the sun it corresponds to lines in the region about 6400–6900 A. Then (8) becomes

$$\frac{d^2J_\nu'}{d\tau^2} = q^2(J_\nu' - J) \quad . \quad . \quad . \quad . \quad (10)$$

These equations must apply when $\eta = 0$ and $J_\nu' = J$. Then (10) and (7) give

$$\frac{d^2J}{d\tau^2} = 0, \quad \frac{dJ}{d\tau} = 3H \quad . \quad . \quad . \quad (11)$$

from which we obtain by integration (with the approximate boundary condition $J = 2H$)

$$J = H(2 + 3\tau) \quad . \quad . \quad . \quad . \quad (12)$$

an equation well known for the integrated radiation, but only holding (as here) for separate frequencies when $J = J_T$. By (10) and (11)

$$\frac{d^2(J_\nu' - J)}{d\tau^2} = q^2(J_\nu' - J) \quad . \quad . \quad . \quad (13)$$

of which the solution is *

$$J_\nu' = J + Ae^{-q\tau} \quad . \quad . \quad . \quad . \quad (14)$$

$$= H(2 + 3\tau) + Ae^{-q\tau}.$$

Hence by (7)

$$H_\nu' = \frac{1}{3(1+\eta)}\{3H - qAe^{-q\tau}\} \quad . \quad . \quad . \quad (15)$$

The value of A is determined by the boundary condition $J_\nu' = 2H'$ at $\tau = 0$. We find

$$A = -2H\eta/(1 + \eta + \tfrac{2}{3}q) \quad . \quad . \quad . \quad (16)$$

Substituting in (15), we have at $\tau = 0$

$$\frac{H_\nu'}{H} = \frac{1 + \frac{2}{3}q}{1 + \eta + \frac{2}{3}q} \quad . \quad . \quad . \quad . \quad (17)$$

which determines the blackening at any point in the line. This formula is equivalent to *I.C.S.* (235·4).

Modern observation is directed towards the width of the line rather than to the blackening at the centre. A standard point to measure is that at which $H_\nu'/H = \frac{1}{2}$, giving the "half-width of the line." Putting $H_\nu'/H = \frac{1}{2}$ in (17), we have

$$\eta = 1 + \tfrac{2}{3}q,$$

whence by (9)

$$(\eta - 1)^2 = \tfrac{4}{3}(1 + \eta)(1 + \epsilon\eta).$$

When $\epsilon = 0$, the result is

$$\eta = 3{\cdot}431,$$

that is to say, the line-absorption at this part of the line is nearly $3\frac{1}{2}$ times the general absorption. The result is not unduly sensitive to collision effects; thus if $\epsilon = 0{\cdot}1$ we find $\eta = 4{\cdot}09$.

The following table shows the absorption-ratio required to produce a given strength of line (when $\epsilon = 0$):—

H_ν'/H.	η.	H_ν'/H.	η.
0·8	0·62	0·4	6·12
0·7	1·16	0·3	12·07
0·6	2·00	0·2	29·5
0·5	3·43	0·1	126·2

* We assume that ε is constant as well as η, so that q is constant.

For very strong absorption $H_\nu'/H \sim 2/\sqrt{(3\eta)}$, the intensity being proportional to the square root of the absorption coefficient (*I.C.S.*, p. 342).

5. If we do not limit ourselves to the case $J_T = J$, the actual value of J_T as a function of τ must be calculated from such knowledge as we have of the temperature at various depths and employed in solving (8). (The internal temperature is, by the usual theory of radiative equilibrium, a known function of the general optical depth, but not of the τ here employed which refers to the immediate proximity of the absorption line.) We shall take the case (treated by Milne according to the Schuster-Schwarzschild approximation *)

$$J_T = a + b\tau \quad . \quad . \quad . \quad . \quad (18)$$

where a and b are constants. There is no mathematical difficulty in treating other forms of J_T since it is merely a question of finding a particular integral of the very simple differential equation (8); but it is scarcely likely that we can have sufficiently detailed knowledge of the actual conditions to justify more than a linear approximation.

By (18) $d^2J_T/d\tau^2 = 0$, and our previous analysis with the substitution of J_T for J applies until we reach (14), which becomes

$$J_\nu' = a + b\tau + Ae^{-q\tau} \quad . \quad . \quad . \quad (19)$$

Then

$$H_\nu' = \frac{1}{3(1+\eta)}(b - qAe^{-q\tau}) \quad . \quad . \quad . \quad (20)$$

The boundary condition $J_\nu' = 2H_\nu'$ determines A, viz.

$$-A = \frac{a(1+\eta) - \frac{2}{3}b}{1 + \eta + \frac{2}{3}q},$$

so that at the boundary

$$H_\nu' = \tfrac{1}{3}(b + aq)/(1 + \eta + \tfrac{2}{3}q) \quad . \quad . \quad . \quad (21)$$

We obtain H from (21) by setting $\eta = 0$, $q = \sqrt{3}$.

$$H = \tfrac{1}{3}(b + a\sqrt{3})/(1 + \tfrac{2}{3}\sqrt{3}),$$

so that

$$\frac{H_\nu'}{H} = \frac{b + aq}{b + a\sqrt{3}} \cdot \frac{1 + \frac{2}{3}\sqrt{3}}{1 + \eta + \frac{2}{3}q} \quad . \quad . \quad . \quad (22)$$

6. As already stated our standard results in § 4 refer strictly to lines formed in a particular part of the spectrum—different for stars of different temperature, but near $\lambda 6500$ for the sun. The use of (22) is to estimate the change involved in considering lines to the red or the blue of the standard position. We show below that the change is appreciable, and the same coefficient of line-absorption will give somewhat wider and deeper lines at the blue than at the red end of the spectrum.

For illustration, we calculate the numerical value of b/a for an atmosphere in which the continuous absorption k is independent of ν.

* *Monthly Notices*, **89**, 7, 1928.

The general radiation of the star is then (as Milne first showed) practically indistinguishable from black-body radiation. Presumably, therefore, it is not far from the actual condition. For k uniform, the internal temperature is given by

$$T^4 = T_0^4(1 + \tfrac{3}{2}\tau) \quad . \quad . \quad . \quad . \quad (23)$$

Also

$$J_T = C/(e^x - 1), \quad x = h\nu/RT \quad . \quad . \quad . \quad (24)$$

From (23) and (24) we find

$$\frac{1}{J_T}\frac{dJ_T}{d\tau} = \frac{3}{8}\frac{xe^x}{e^x - 1}\cdot\frac{1}{1 + \frac{3}{2}\tau} \quad . \quad . \quad . \quad (25)$$

In stars of solar type we are concerned with fairly large values of x for which $e^x/(e^x - 1)$ can be set equal to unity. Hence by (18) we have (at $\tau = 0$)

$$b/a = \tfrac{3}{8}x \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The following results relating to a star of about solar type are obtained from (22) and (26). The first two lines of the table illustrate low absorption coefficients appropriate to the study of the half-width of the spectral line; the last line is an example of a high absorption coefficient concerned in the deeper part of the line. The blackening (H_ν'/H) is tabulated and the extent of its variation with wave-length is seen from the table. The column $x = 4$ is the "standard case" treated in § 4.

Blackening (H_ν'/H) in Different Parts of the Spectrum.

Wave-length. x.	3000. 8.	4000. 6.	6000. 4.	8000. 3.
$\eta = 3$	·467	·490	·525	·549
$\eta = 4$	·413	·437	·464	·497
$\eta = 74$	·096	·110	·129	·143

7. The obvious point in which the actual stellar conditions will differ widely from those adopted in § 4 is in regard to the constancy of η at different depths. Even if the material responsible for the absorption line has constant concentration η will not be constant, because according to our present belief k increases with depth (nearly proportionately to the pressure). We shall later (§ 10) investigate a case in which η varies continuously with depth. Here we take a discontinuous case, viz. a layer of selectively absorbing atoms limited above and below, η being constant within the layer and zero outside it.*

Let the layer be between the depths τ_1 and τ_2, so that between these limits q, $\eta = q_1$, η_1, and outside the limits $q = q_0 = \sqrt{3}$ and $\eta = 0$. Let

$$z = \int_0 q d\tau + \tanh^{-1}\tfrac{1}{2}\sqrt{3} \quad . \quad . \quad . \quad (27)$$

* The solutions for the simpler problem of a layer limited on one side only (above or below) were obtained in *I.C.S.*, §§ 236, 237.

Then the solutions of (10) appropriate to the three regions are found to be

$$\left.\begin{aligned} J_\nu' &= J - A\cosh z \\ H_\nu' &= H - \tfrac{1}{3}q_0 A\sinh z \end{aligned}\right\}\text{surface layer}$$
$$\left.\begin{aligned} J_\nu' &= J - B\sinh(z-\beta) \\ (1+\eta_1)H_\nu' &= H - \tfrac{1}{3}q_1 B\cosh(z-\beta) \end{aligned}\right\}\text{middle layer} \qquad (28)$$
$$\left.\begin{aligned} J_\nu' &= J - Ce^{-z} \\ H_\nu' &= H + \tfrac{1}{3}q_0 Ce^{-z} \end{aligned}\right\}\text{deep layer}$$

These satisfy the boundary condition $J' = 2H'$ at $\tau = 0$, and the condition of convergence at great depth. The four constants A, B, C, β are to be found from the continuity of J', H', at the two transitions. This gives

$$\begin{aligned} B\sinh(z_1-\beta) &= A\cosh z_1, \\ B\cosh(z_1-\beta) &= uA\sinh z_1 - 3H\eta_1/q_1, \\ B\sinh(z_2-\beta) &= Ce^{-z_2}, \\ B\cosh(z_2-\beta) &= -4Ce^{-z_2} - 3H\eta_1/q_1, \end{aligned}$$

where

$$u = q_0(1+\eta_1)/q_1 = \sqrt{(1+\eta_1)}/\sqrt{(1+\epsilon\eta_1)} \quad . \qquad (29)$$

These lead to

$$\tanh(z_1-\beta) = \frac{1-\cosh(z_2-z_1) - u\sinh(z_2-z_1)}{\sinh(z_2-z_1) + u\cosh(z_2-z_1) + u\tanh z_1} \qquad (30)$$

$$B\{\cosh(z_2-\beta) + u\sinh(z_2-\beta)\} = -3H\eta_1/q_1 \quad . \qquad (31)$$

Thus β is found, and then B ; the other constants A and C easily follow.

At the surface $\tau = 0$, $\sinh z = \sqrt{3}$, and the second equation of (28) reduces to

$$H_\nu' = H - A \quad . \quad . \quad . \quad . \qquad (32)$$

so that the constant A determines the observed strength of the line.

It is of interest to note the results for a very thin but opaque layer such that $\eta_1\delta\tau$ is finite. In this case δz is small but $u\delta z$ is finite. The formulæ reduce to

$$\beta - z_1 = \delta z/(1+\tanh z_1), \qquad z_2 - \beta = \tanh z_1\delta z/(1+\tanh z_1),$$

and

$$\frac{A}{H} = \frac{\sqrt{3}\,.\,u\delta z}{e^{z_1} + \sinh z_1\,.\,u\delta z} = \frac{3\eta_1\delta\tau}{e^{z_1} + \sqrt{3}\sinh z_1\,.\,\eta_1\delta\tau}.$$

Even if the layer is at the surface of the star this is entirely different from the loss $\eta_1 d\tau$ which so-called elementary considerations, based on our experience of absorption under conditions remote from thermodynamic equilibrium, would suggest. (At the surface $z_1 = 1{\cdot}32$ in accordance with (27).)

8. The most interesting application of § 7 is to trace the growth of an absorption line. Three examples are shown in the diagram. The abscissæ (from right to left) indicate optical depth τ below the surface. Vertical lines in each figure mark the limits of the selectively absorbing

material (distinguished by shading at the foot). The full curve is the graph of H_ν'/H which measures the intensity of the outward radiation; the broken curve is the graph of J_ν'/J which measures the density of the radiation in all directions, compared with the equilibrium density just outside the line. Owing to the boundary condition (zero inward radiation) the two curves always meet at the boundary.

The full curve H_ν'/H represents the growth of the line—if any meaning can properly be attached to the intensity of the line within the atmosphere. Practically the whole growth of the line occurs *beneath* the

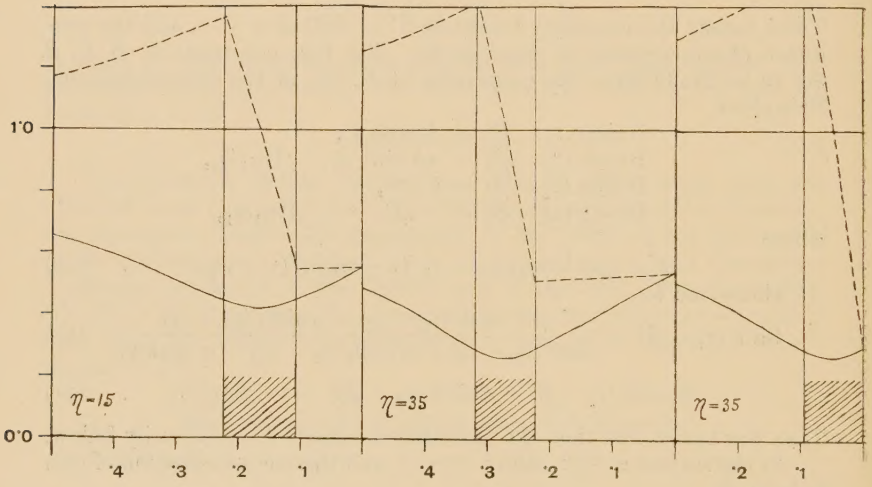

FIG. 1. FIG. 2. FIG. 3.

Formation of Absorption Lines.

H_ν'/H ——; J_ν'/J - - - - -.

absorbing layer. The line deepens a little in the lower half of the layer, but there is a recovery in the upper half, so that there is very little net deepening of the line between the bottom and top of the layer. Above the layer the line fills up rapidly.

The complicated behaviour of J_ν'/J is best ascertained from the diagram. Perhaps the most curious feature is that during the passage through the atmosphere above the absorbing layer the recovery from the minimum value is barely appreciable.

The *modus operandi* appears to be that the radiation trying to pass upwards gets heaped up beneath the opaque layer. This causes a gradient which checks the outward flow. It is like a queue passing through a narrow barrier; onward progress is arrested long before the obstruction itself is reached.

To the left of each figure both curves continue to approach the ordinate 1 asymptotically; evidently the disturbed conditions extend a very long way.

A comparison of the three diagrams shows the importance of proximity to the surface. The absorbing layer, which in fig. 2 lies at a moderate depth, is moved up to the surface in fig. 3. The curve for H_ν'/H is scarcely altered, but, "the recovery belt" being eliminated, we see a much stronger line. Comparison of figs. 1 and 2 shows how the deeper absorption in the latter is neutralised by a longer opportunity of recovery, and the observed lines are of practically equal strength.

The following table gives some of the calculated points of the curves. (It is added because various small differences and asymmetrical features cannot be ascertained from the diagram.) The minimum of H_ν'/H necessarily occurs at $z = \beta$. We have taken $\epsilon = 0$.

Fig. 1 ($\eta = 15$).				Fig. 2 ($\eta = 35$).				Fig. 3 ($\eta = 35$).			
z.	τ.	H_ν'/H.	J_ν'/J.	z.	τ.	H_ν'/H.	J_ν'/J.	z.	τ.	H_ν'/H.	J_ν'/J.
1·317	0	·552	·552	1·317	0	·538	·538	..	..	..	..
z_1	·106	·450	·545	z_1	·221	·295	·510	z_1	0	·303	·303
β	·166	·417	1·0	β	·270	·262	1·0	β	·048	·269	1·0
z_2	·222	·444	1·362	z_2	·317	·290	1·417	z_2	·096	·296	1·533
2·782	·5	·656	1·170	3·017	·5	·483	1·256	2·670	·3	·505	1·295

9. The solution for a layer of absorbing material extending from the surface to a depth τ_2 is included in § 7, but it is easier to obtain the results for this simpler case independently (as in *I.C.S.*, § 237) than to adapt the general solution. The strength of the line is found to be given by

$$\frac{H - H_\nu'}{H} = \frac{\sqrt{3}u\eta}{2(1+\eta)} \frac{u \sinh \delta z + \cosh \delta z - 1}{(\frac{1}{2}\sqrt{3}u^2 + 1)\sinh \delta z + (\frac{1}{2}\sqrt{3} + 1)u \cosh \delta z} \qquad (33)$$

where $\delta z = z_2 - z_1 = q\tau_2$.

This is especially important because it provides an answer to the question, What part of the atmosphere are we chiefly concerned with in the problem of line-absorption? For example, for any given value of η_1 we can by trial and error find a value $\delta z'$ which makes the second factor in (33) nine-tenths of its full value for $\delta z = \infty$. Then we may fairly say that the absorption line is determined by the state of affairs above optical depth $\tau' = \delta z'/q$, opacity below τ' having comparatively little effect. In common terminology this corresponds to fixing approximate limits of the reversing layer; it is obviously of the first importance in all discussions to know roughly the level of the region that is concerned.

Taking first $\eta = 3$, giving a weak line ($H_\nu'/H = ·52$), I find that approximately $\frac{9}{10}$ of the full strength is obtained when $\delta z' = 2$. The corresponding optical depth τ is 0·58.

If $\eta = 24$, giving a moderately strong line ($H_\nu'/H = ·22$), I find $\delta z' = 1·4$, and hence $\tau = 0·16$.

In problems regarding the half-width of a line the value of η concerned is a little greater than 3, so that for this purpose the lower limit

of the reversing layer is at optical depth about 0·5. For stronger parts of the line it is at a much less depth.

10. In order further to supplement the standard solution of § 4, we now take an example of the ratio of selective to general absorption (η) varying continuously with depth. Treating η as variable, but neglecting ϵ, we have in place of (10)

$$\frac{d}{d\tau}\left(\frac{1}{1+\eta}\frac{dJ_\nu'}{d\tau}\right) = 3(J_\nu' - J) \quad . \quad . \quad . \quad (34)$$

I shall treat the case in which the variation of η is according to the formula

$$\frac{1}{1+\eta} = \alpha + \beta\tau \quad . \quad . \quad . \quad . \quad (35)$$

If β is positive this means that η decreases with depth, as would happen, for example, if k increases with pressure (as generally supposed) and l is unchanged. (For the difficulty which might arise through η becoming negative at great depths, see § 11.) Accordingly

$$\frac{d}{d\tau}\left\{(\alpha+\beta\tau)\frac{dJ_\nu'}{d\tau}\right\} = 3(J_\nu' - J) \quad . \quad (36)$$

Using (12) we find a particular integral

$$J_\nu' = H(2 + \beta + 3\tau).$$

To find the complementary function set

$$y = 12(\alpha + \beta\tau)/\beta^2 \quad . \quad . \quad . \quad . \quad (37)$$

Then the equation becomes

$$\frac{d}{dy}\left(y\frac{dJ_\nu'}{dy}\right) = \tfrac{1}{4}(J_\nu' - J).$$

Now put $x = \sqrt{y}$. We obtain

$$\frac{d^2J_\nu'}{dx^2} + \frac{1}{x}\frac{dJ_\nu'}{dx} - J_\nu' = -J.$$

The left side is a Bessel equation of order zero for imaginary argument, of which the solutions are denoted by $I_0(x)$ and $K_0(x)$. $I_0(x)$ is inapplicable since it increases exponentially for large values of x. Hence the complete solution of (36) for our purpose is

$$J_\nu' = H(2 + \beta + 3\tau) + AK_0(x) \quad . \quad . \quad . \quad (38a)$$

Hence by (7), after a little reduction,

$$H_\nu' = H(\alpha + \beta\tau) + A\sqrt{\tfrac{1}{3}(\alpha+\beta\tau)}K_0'(x) \quad . \quad . \quad (38b)$$

A is determined by the condition $J_\nu' = 2H_\nu'$ when $\tau = 0$, giving

$$-A = \frac{(2 + \beta - 2\alpha)H}{K_0(x_0) - 2\sqrt{(\tfrac{1}{3}\alpha)}K_0'(x_0)} \quad . \quad . \quad . \quad (39)$$

Hence at the boundary by (38*b*)

$$\frac{H_\nu'}{H} = \frac{\alpha - \sqrt{(\frac{1}{3}\alpha)}(2+\beta)K_0'/K_0}{1 - \sqrt{(\frac{1}{3}\alpha)}\,.\,2K_0'/K_0} \quad . \quad . \quad (40)$$

the argument x_0 of K_0'/K_0 being $\sqrt{(12\alpha/\beta^2)}$.

Tables of K_0 and $K_1(= -K_0')$ are given in Watson's *Bessel Functions*. For large values of x we might use an approximation derived from the asymptotic expansions, viz.

$$-\frac{K_0'(x)}{K_0(x)} = \frac{8x+3}{8x-1},$$

which begins to be reasonably adequate when $x > 3$. We are, however, often concerned with smaller values.

11. It may be objected that the adopted law (35) requires that beyond a certain depth η should become negative, which is physically impossible. This scarcely matters because for values of α and β which we are likely to be interested in, the level is too deep to enter in an important way into the results. The objection can, however, be removed at the expense of a little additional algebra. We take

$$1/(1+\eta) = \alpha + \beta\tau \qquad (\alpha + \beta\tau < 1) \quad . \quad . \quad (41)$$
$$= 1 \qquad (\alpha + \beta\tau > 1).$$

Then it is a question of joining on a solution

$$J_\nu' = H(2+\beta+3\tau) + AK_0(x) + BI_0(x)$$

to a solution

$$J_\nu' = H(2+3\tau) + Ce^{-\tau\sqrt{3}}$$

at the depth $\alpha + \beta\tau = 1$, or $x = \sqrt{(12/\beta^2)}$. Continuity of J_ν' and H_ν' at the junction gives the condition

$$-H\beta = A\{K_0(x_1) + K_0'(x_1)\} + B\{I_0(x_1) + I_0'(x_1)\} \qquad (x_1 = \sqrt{(12/\beta^2)}) \quad . \quad (42)$$

The surface condition replacing (39) is

$$-H(2+\beta-2\alpha) = A\{K_0(x_0) - 2\sqrt{(\tfrac{1}{3}\alpha)}K_0'(x_0)\} + B\{I_0(x_0) - 2\sqrt{(\tfrac{1}{3}\alpha)}I_0'(x_0)\} \qquad (x_0 = \sqrt{(12\alpha/\beta^2)}) \quad (43)$$

And at the surface

$$H_\nu' = H\alpha + \sqrt{\tfrac{1}{3}\alpha}(AK_0'(x_0) + BI_0'(x_0)) \quad . \quad . \quad (44)$$

The blackening H_ν'/H is then found by eliminating A and B from (42), (43), and (44), the elimination being most conveniently performed after the numerical values of the Bessel functions have been inserted.

It may be noted that the formula (40) as it stands does not preclude values of H_ν'/H greater than unity, *i.e.* an emission line. The condition $H_\nu' > H$ gives

$$-\beta K_0'/K > (1-\alpha)/\sqrt{(\tfrac{1}{3}\alpha)},$$

and, since $-K_0'/K > 1$, this is satisfied if

$$\beta > (1-\alpha)/\sqrt{(\tfrac{1}{3}\alpha)}.$$

The result is presumably spurious, since the mechanism here under discussion could not conceivably strengthen the outflowing radiation. No doubt the explanation is that the large value of β brings the zero of η close to the surface; and (in default of corrections according to the method of this section) we obtain an emission line, because we are inadvertently attributing to the line-frequency an average absorption coefficient less than that of the continuous spectrum.

12. I give below three examples of the calculation of line-intensity for variable η. In the first two the result H_ν'/H is obtained directly from (40). In the third I obtained from (40) the value 0·465; but in this case there seemed to be a possible risk in neglecting the correction described in § 11. I therefore recalculated it from § 11 and found the more accurate value 0·450.

In order to judge the amount of variation of η postulated in the three examples, its values are given at five depths in the important region of the star. Then follows the deduced intensity in the line (H_ν'/H). The next line gives $\bar{\eta}$, *i.e.* the uniform value of η which would yield the same line-intensity. Finally, $\bar{\tau}$ is given by $1/(1+\bar{\eta}) = \alpha + \beta\bar{\tau}$; it is the depth at which the actual variable opacity is equal to the equivalent uniform opacity.

α	0·1	0·05	0·05
β	0·5	0·25	1·0
η at $\tau=0·0$	9	19	19
,, 0·1	5·7	12·3	5·7
,, 0·2	4	9	3
,, 0·3	3	7	1·9
,, 0·4	2·3	5·7	1·2
H_ν'/H	0·452	0·296	0·450
$\bar{\eta}$	4·49	12·42	4·55
$\bar{\tau}$	0·16	0·098	0·13

The values of $\bar{\tau}$ are very much what might have been anticipated from our conclusions in § 9 as to the limits of the reversing layer.

A term ϵ representing superelastic collisions was included in the general theory but has usually been dropped in the detailed solutions. One reason for not pursuing it is that ϵ is obviously very variable with depth, and there is not much satisfaction in finding solutions which imply constant ϵ. In the present section it may, if desired, be included quite naturally. Since (for constant temperature) ϵ and k are both proportional to the pressure, $\epsilon l/k$ ($= \epsilon\eta$) has no particular tendency to change with depth. Hence it is not unfair to treat $(1+\eta)$ as variable in (7) and adopt a constant $(1+\epsilon\eta)$ in (6). The analysis of §§ 10, 11 then applies with the slight modification due to the insertion of this

constant factor where required. The work has to be gone over again in detail, because α and β appear sometimes with the modifying factor and sometimes without. I find the result corresponding to (40),

$$\frac{H_\nu'}{H} = \frac{\alpha - \sqrt{(\frac{1}{3}\alpha C)(2 + \beta/C)}K_0'/K_0}{1 - \sqrt{(\frac{1}{3}\alpha C)}2K_0'/K_0} \quad . \quad . \quad (45)$$

the argument of K_0'/K_0 being $\sqrt{(12\alpha C/\beta^2)}$. $C = 1 + \epsilon\eta$.

13. In general arguments about absorption lines reference is often made to the number of atoms in an excited state, and it is useful to examine our results from this point of view.* Let n' be the ratio of the number of atoms in an upper state to the number in the lower state, and let n be the corresponding ratio in thermodynamic equilibrium at the temperature concerned. In the case we have hitherto been dealing with we have (neglecting superelastic collisions)

$$\frac{n'}{n} = \frac{J_\nu'}{J_T} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

the number of excited atoms being proportional to the density of the exciting radiation (so long as n and n' are small).

This, however, is rather a special case, and in general the number of atoms in an excited state depends not only on the spectral line under consideration, but on the lines corresponding to many other transitions. Accordingly, as a first step we shall set $n' = fn$ and examine the intensity of the spectral line for arbitrarily prescribed values of f.

The last three terms in (4), representing emission, will be modified accordingly. It is easily seen that we must replace them by $(k + fl)J_T$. (The emission in thermodynamic equilibrium must be $(k + l)J_T$ in order to balance the absorption, and with $n' = fn$ the line part of the emission is altered f-fold.) Hence the fundamental equation is

$$\cos\theta\frac{dJ_\nu'(\theta)}{\rho dr} = -(k + l)J_\nu'(\theta) + (k + fl)J_T \quad . \quad (47)$$

and in place of (6) and (7) we have

$$\frac{dH_\nu'}{d\tau} = (1 + \eta)J_\nu' - (1 + f\eta)J_T \quad . \quad . \quad . \quad (48)$$

$$\frac{dJ_\nu'}{d\tau} = 3(1 + \eta)H_\nu' \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Multiplying (49) by $1/\sqrt{3}$ and adding to (48),

$$\frac{dH_\nu'}{d\tau} - \sqrt{3}(1 + \eta)H_\nu' + \frac{1}{\sqrt{3}}\left(\frac{dJ_\nu'}{d\tau} - \sqrt{3}(1 + \eta)J_\nu'\right) = -(1 + f\eta)J_T \quad (50)$$

Let

$$\sigma = \sqrt{3}(1 + \eta) \quad . \quad . \quad . \quad . \quad (50a)$$

* It now becomes necessary to consider the strength of the line as a whole, and our results refer to this. For the possibility that the wings of the line may behave differently, see § 15.

Then (50) becomes integrable on multiplying by $e^{-\sigma\tau}$, giving

$$(H_\nu' + J_\nu'/\sqrt{3})e^{-\sigma\tau} = -\int(1 + f\eta)e^{-\sigma\tau}J_T d\tau.$$

Hence, at the surface where $\tau = 0$, $J_\nu' = 2H_\nu'$,

$$(1 + 2/\sqrt{3})H_\nu' = \int_0^\infty (1 + f\eta)e^{-\sigma\tau}J_T d\tau \quad . \quad . \quad (51)$$

The expression of H_ν' as an integral of this type is a method that has frequently been employed by Milne.

14. In the diagrams illustrating § 8, J_ν'/J has a low value near the surface so that n'/n is abnormally low; evidently the consequent reduced emission is partly responsible for the darkness of the line. But it would be a mistake to suppose that no spectral line is formed if the number of excited atoms is maintained at the value for thermodynamic equilibrium. This case ($f = 1$) is covered by the investigation in §§ 3, 4, by setting $\epsilon = 1$; for the fundamental equations (4) and (47) then become the same.* This gives $q = (1 + \eta)\sqrt{3} = \sigma$, and by (17)

$$\frac{H_\nu'}{H} = \frac{1 + \eta + \frac{1}{2}\sqrt{3}}{(1 + \eta)(1 + \frac{1}{2}\sqrt{3})} \quad . \quad . \quad . \quad (52)$$

For example, if $\eta = 11$, the condition $n' = n$ leads to $H_\nu'/H = 0{\cdot}574$, as compared with $0{\cdot}312$ when the conditions are as previously assumed.

If f is a constant we see from (51) that H_ν' is multiplied in the ratio

$$g = (1 + f\eta)/(1 + \eta) \quad . \quad . \quad . \quad (53)$$

as compared with the value (52) for $f = 1$.

The formulæ (52) and (53) bring out an important point, viz. that current investigations of line-intensity relate to a very special case in which the number of atoms in an excited state is conditioned wholly or mainly by the line that is being studied. If we admit a more general case in which the number of excited atoms depends on other causes the results are altogether different, and in fact the darkness of the line has very little to do with its absorption coefficient. This will be seen from a numerical example.

Taking first $f = 1$, let us increase η from 2 to ∞. The line deepens only from $0{\cdot}691$ to $0{\cdot}536$, a change which would be counteracted by increasing f 30 per cent. It seems fair to say that the intensity of a line is mainly dependent on f and gives very little indication of its own absorption coefficient.

An indirect influence of the absorption coefficient is probably more important. When f is not constant the line-intensity depends on its average value, weighted in a manner indicated by (51), viz. with a factor $e^{-\sigma\tau}$. For large absorption coefficients the weight is thus thrown very near the surface. The J_ν's of the various lines of the spectrum are presumably decreasing as the surface is approached; consequently

* The case $\epsilon = 0$ is called by Milne "monochromatic radiative equilibrium," and the case $\epsilon = 1$ "local thermodynamic equilibrium."

f is likely to change rapidly near the surface, and this weighting effect of η is likely to play an important part.

15. So far as I know, no progress has been made in the theory of line-intensity of ordinary spectra in which alternative transitions are simultaneously concerned. I have tried the simplest case of an atom with 3 states connected by 3 transitions, but found it intractable. It was not even possible to guess what kind of intensities might be expected. I do not think that any existing prediction or expectation of the relative intensities of, say, the dark Balmer lines in stellar spectra is deserving of the slightest credence. To apply the theory of §§ 3–12 would be absurd. The atomic absorption coefficients given by the quantum theory are based on the probabilities calculated for the various alternative transitions; we cannot go on to apply these to a theory which excludes alternative transitions.

It might seem more hopeful to deduce from the observed intensities of the lines the factor f, giving the relative numbers of atoms in different states, and, combining the ratios deduced from various lines, to build up a complete scheme of distribution of excited states. But we cannot combine f's which each refer to a different depth in the star depending on the η of the line involved. If we ignore this variation of depth, our "approximation" is virtually equivalent to neglecting all differences of absorption coefficients (since their other influence is a very minor one). This would seem too revolutionary a method of treating stellar spectra!

If Unsöld's theory permits a wider application of the simple theory, it must be by postulating a difference of condition between the wing of a line and the main central absorption. I do not think that there can be any difference in the nature of the processes concerned; but it is possible that an approximation obviously invalid for the centre may be adequate for the wing. There seems to be a general idea that when an atom absorbs light wide of the line-centre the re-emission follows very quickly (quicker than for a central excitation). If so, excitation to a still higher level has little chance of occurring and one cause of interlocking of spectral lines is reduced. It is also necessary to assume that some kind of "memory" of its former transition can last the brief time of wing excitation, so that the atom goes back to its first state; whereas after the longer time of central excitation the route of descent is a matter of chance. Further, it must be supposed that wing-emission rarely happens except as a sequel to wing-absorption. If these conditions are satisfied the formation of the wings of lines becomes for each line an isolated problem and treatment according to the methods of §§ 2–12 is justified. If "half-width" measurements can be considered to be far enough in the wing, they may be interpreted according to the formulæ there deduced. If wing excitation is of brief duration superelastic collisions will be negligible; it is in fact scarcely consistent to retain these in our investigation if we neglect the influence of other lines belonging to the same spectrum.

I think that the assumptions in the foregoing paragraph are made implicitly in all current attempts to interpret line-contours, except

sometimes in problems restricted to doublets (*e.g.* H and K) and multiplets where the absence of a transverse transition simplifies matters. I believe that at any rate not all the assumptions have been made inadvertently ; but I am not sure that an adequate examination has been made whether collectively they are in accordance with present physical theory or experiment. The subject is too dangerous to tackle lightly, since it involves in a rather unusual degree a mixture of the classical atom, the Bohr atom, and the Schrödinger atom in their most (apparently) contradictory aspects. From those physicists whom I have consulted I have been unable to extract a coherent view.

Collision and the Origin of Rotation in the Solar System.
By Harold Jeffreys, D.Sc., F.R.S.

The theory that the solar system originated from the disruption of the sun by a passing star appears to meet many of the requirements of a theory of the origin of the solar system. In its early form, given by Buffon,* the disturbing body was called a "comet," but it was indicated that it was supposed to be an object of considerable mass. The encounter was treated as a material collision. The rotation of the sun was attributed to shearing forces during collision. The theory was revived by Bickerton,† who suggested a means of meeting an objection raised by Laplace connected with the circular form of the planets' orbits. T. C. Chamberlin ‡ replaced the material collision by a tidal encounter without actual contact, and the mechanics of such tidal encounters has been developed by Jeans and the present writer.

A purely tidal theory has to face difficulties of two types. First, if we denote the masses of the sun and star by M and M′, the radius of the sun by a, and the least distance between the centres by R_0, a necessary condition for tidal disruption to be possible is that M'/R_0^3 shall be at least comparable with M/a^3. The mass of the star has to be such that if it was expanded to fill a sphere of radius R_0 the density would be comparable with that of the sun. Except for very close approaches, amounting practically to grazing collision, the mass of the star must be a larger multiple of that of the sun than astronomical data indicate to be at all frequent.

The other and the more serious difficulty concerns the rotations of the sun and planets. Rotation in a fluid is a difficult thing to produce. If we denote the Cartesian co-ordinates of position by (x, y, z) and the corresponding velocities by (u, v, w), and define the circulation as the integral

$$\Omega = \int u dx + v dy + w dz \quad . \quad . \quad . \quad (1)$$

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* *Natural History*, 1750. English translation by Smellie, 1812.
† *Trans. New Zealand Institute*, 1880, pp. 149–159.
‡ *Astrophys. Journ.*, 1901, pp. 309–333.

taken around any circuit of particles in the fluid, then according to a fundamental theorem of classical hydrodynamics Ω remains the same for all time if the circuit is always taken through the same particles. So long as the conditions for the validity of this theorem are satisfied, there is no way of generating rotation by the movement of solid boundaries or by bodily gravitational forces. In a sphere or cylinder of radius a rotating with angular velocity ω, the circulation around the margin of its section is $2\pi a^2\omega$.

If the rotations of the sun and most of the planets in planes near the ecliptic are to be explained, we must refer them to some cause connected with the origin of the system. We should, if the circulation theorem is valid, expect all to have the same rate of rotation, apart from variations in radius due to condensation, and this rotation would be that of the sun *before* the encounter. There is no reason why this should have been anywhere near the plane of the ecliptic. Further, the periods of rotation, excluding those probably affected by tidal friction, range from 10 hours for Jupiter to 25 days for the sun. Jupiter and the sun have about the same density, and, since variations of the radius of a condensation would affect its density, we must suppose that Jupiter has about the same radius as the matter that formed it had when in the sun before ejection. No means of accounting for the difference of rotation is provided.

The tidal theory, in its present form, attributes the rotation of the sun to matter ejected from it and falling back into it after having acquired a transverse deflexion from the attraction of the star; and those of the planets to matter falling back into them after they had been tidally broken up by the sun on their first perihelion passages. Now the angular momentum of a particle of mass m is

$$h = m(f\mathrm{M}l)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (2)$$

where f is the constant of gravitation, M the mass of the primary, and l the semi-latus rectum of the orbit. For an elongated orbit l is nearly twice the least apsidal distance, and for a collision the least apsidal distance must be less than the radius of the primary. Hence for a particle reabsorbed by the sun

$$h<m(2f\mathrm{M}a)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (3)$$

$$<m(4 \times 10^{18}) \quad . \quad . \quad . \quad . \quad (4)$$

in c.g.s. units; for $f = 6{\cdot}6 \times 10^{-8}$, $\mathrm{M} = 2 \times 10^{33}$ gm., $a = 7 \times 10^{10}$ cm. The sun's angular momentum is about 10^{49} c.g.s. units. Hence the mass that fell back into the sun must have been at least 2×10^{30} gm., about the mass of Jupiter.

When we apply the same argument to the rotation of Jupiter, we find

$$h<m(4 \times 10^{16}) \quad . \quad . \quad . \quad . \quad (5)$$

The angular momentum of rotation is $5{\cdot}5 \times 10^{45}$ c.g.s. Hence the mass reabsorbed was at least $1{\cdot}4 \times 10^{29}$ grams, or 0·07 of the mass of Jupiter. The total mass of the existing satellites is about $\frac{1}{6000}$ of the

mass of Jupiter. Four hundred times as much matter must have been reabsorbed as escaped. Similar considerations apply to the other great planets and to Mars. The earth, Venus, and Mercury need not be considered on account of the probable influence of tidal friction on their rotations since they were formed.

The above result for the sun might perhaps be acceptable. That for Jupiter requires that, of the matter ejected by that planet, only $\frac{1}{400}$ escaped reabsorption on the first return to pericentre. This is very questionable. The reabsorbed matter has been assumed, for numerical purposes, to have had a pericentral distance equal to the radius of the planet. Yet the amount with an original pericentral distance greater than the radius of the planet is only $\frac{1}{400}$ of the whole; insignificant room is left for the departures from the mean value that must inevitably have existed on the tidal theory. If we adopt a smaller mean pericentral distance for the returning matter, the angular momentum per unit mass is also diminished and the fraction supposed to have missed the primary on return is made even smaller.

Distension of the great planets at the time makes a slight improvement. But the angular momentum required is the actual angular momentum, calculable from present data; that capable of being carried by a given mass on returning is proportional only to the square root of the radius of the planet. To decrease the reabsorbed mass to $\frac{1}{400}$ of the value just given we should have to suppose the primitive radius of the planet $1{\cdot}6 \times 10^5$ times as great as at present—an utterly impossible distension.

Suggested explanations of the rotations of the great planets and Mars, therefore, seem to meet serious objections.

In the circumstances it appears to be desirable to bestow some attention on the older theory, according to which the encounter between the sun and the star was a material collision and not merely a close approach. All the forces available on the tidal theory are still available when there is actual collision. The transverse deflexion of ejected matter would lead to elliptic orbits in the same way. Lighter constituents lost by the ejected matter would go to form a resisting medium in the same way, and the later development of the system is provided for. But we have a significant difference. The velocity of escape of a particle from the sun's surface is 450 km./sec. The relative velocity of the sun and a star of comparable mass when they collided would be of this order. Two fluid masses would come into contact, and there would be discontinuities of this order in the velocities on the two sides of the surface of separation. These are just the conditions required to endow a mass with rotation. The viscous forces, which are ignored in classical hydrodynamics, and are usually negligible in fluid that has not passed near a surface of discontinuity, become dominant in fluid that has passed near one, and spread the discontinuity in velocity out through a finite thickness. In such a layer the velocities would range all the way from the velocity of the sun to that of the star. If, by hypothesis, the star and the sun separated again, the layer of transition would therefore be stretched out all the way

from the sun to the star, and such a filament would be formed as the tidal theory supposes. The mechanism of the rupture of the filament described by Jeans is still available. But in addition the shearing motion in the layer of transition constitutes a circulation, and a qualitative explanation of the rotation of the planets is provided.

Full investigation of the impact of two stellar masses is clearly hopeless. Gravity waves of finite amplitude on a plane sheet of perfect liquid are on the verge of mathematical tractability, and here we have the complications of spherical boundaries, compressibility, viscosity, and the peculiarities of stellar matter. But an outline may be attempted. The two bodies are supposed to have collided with a velocity u near 400 km./sec., and the surface of separation to have approached the centre of the sun within a distance about $\frac{1}{3}a$, where a is the radius of the sun. The collision lasted a time t of the order of a/u. Taking a to be 7×10^5 km., t is 1750 seconds, or about half an hour. The time taken for a gravity wave to travel around the sun would be of the order of $2\pi a/(ga)^{\frac{1}{2}}$, where g is the intensity of surface gravity on the sun. But our adopted value of u is nearly $(ga)^{\frac{1}{2}}$ and the time is $2\pi a/u$ or $2\pi t$. The velocity of compressional waves in a gaseous sphere is comparable with that of gravity waves around the boundary, and the time a compressional wave would take to pass through the sun and back would be about $4t$. On account of the factors 2π and 4 it seems probable that the effects would be concentrated near the interface so long as contact lasted. Violent oscillations would, of course, be set up, but it seems possible that the oscillations of the whole masses would not interfere seriously with the course of the collision; waves sent out would not return till after the collision was over.

The viscosity in a star has been shown by Jeans to be mainly radiative in character; * in the sun the effective coefficient of viscosity ν (Jeans's κ_R/ρ) is about 13 cm.^{2}/sec. In time t, if the transfer of vorticity is by pure viscosity, the thickness d of the layer affected would be of order $(\nu t)^{\frac{1}{2}}$ or 150 cm. But this thickness d makes ud/ν of order 5×10^8. By Osborne Reynolds's criterion the pure viscous flow is replaced by turbulent flow if ud/ν exceeds a number of the order of 1000. The actual transfer would therefore be by eddies. In these circumstances the frictional force per unit area is about $0{\cdot}002\ \rho u^2$, where ρ is the density and u the difference of velocity. The coefficient 0·002 is a pure number determined by experiment. The momentum communicated in time t is therefore $0{\cdot}002\rho u^2 t$. But $ut = a$, and the momentum is enough to give a velocity comparable with u to a thickness of order $0{\cdot}002a$. It follows that the mass of the layer of the sun that acquired a velocity of the order of the velocity of the star was of the order of $\frac{1}{500}$ of the mass of the sun. But this was the matter that trailed after the star when it departed and formed the planets. The total mass of the planets is actually comparable with the calculated amount.

Consider next the angular momentum of the layer about an axis through its centre of mass and perpendicular to the general motion. It

* *M.N.R.A.S.*, **86**, 328–335, 444–457, 1926.

would be of order $\pi\rho a^2 \cdot d^2u$, where d is now the thickness of the layer accelerated, and equal to $\frac{1}{500}a$. The mass of the layer would be of order $\pi\rho a^2 d$, and its volume $\pi a^2 d$. If it was collected into a sphere of radius b we should have

$$b^3 = \tfrac{3}{4}a^2 d \quad . \quad . \quad . \quad . \quad (6)$$

The final density being supposed the same as the initial one, the moment of inertia of the sphere would be $\frac{2}{15}\pi\rho b^5$, and its angular momentum when it rotated with angular velocity ω was $\frac{8}{15}\pi\rho b^5\omega$. Hence

$$\tfrac{2}{15}\pi\rho b^5\omega = \pi\rho a^2 d^2 u \quad . \quad . \quad . \quad . \quad (7)$$

which leads to

$$\omega = \frac{5}{2}\frac{u}{(500)^{\frac{1}{3}}a} = \frac{2\times 10^{-4}}{\text{sec.}} \quad . \quad . \quad . \quad (8)$$

The period of rotation of the resulting body, if the whole were collected into one sphere, would be of the order of $2\pi/\omega$ or 3×10^4 seconds, say 8 hours, which agrees with the actual periods of rotation of the great planets.

The total momentum communicated to the layer was of order $\frac{1}{500}\rho ua$ per unit area, and its moment about the centre of the sun $\frac{1}{500}\pi\rho ua^4$. If a moderate fraction of the accelerated matter stayed behind, or else fell back into the sun, keeping its angular momentum, the angular velocity of the sun would be Ω, given by

$$\tfrac{8}{15}\pi\rho a^5\Omega = \tfrac{1}{500}\pi\rho ua^4 \quad . \quad . \quad . \quad . \quad (9)$$

or

$$\Omega = \frac{3}{800}\frac{u}{a} = \frac{2\times 10^{-6}}{\text{sec.}} \quad . \quad . \quad . \quad (10)$$

The period of rotation would be $2\pi/\Omega$ or about 30 days, in rough agreement with the actual rate of rotation of the sun.

These numerical estimates are, of course, orders of magnitude. All rest on the assumption that the area of contact had a radius comparable with the radius of the sun ; glancing collisions are excluded. Also the collision could not be head on, for then the two bodies could not have slid over each other through a distance comparable with their radii, as has been assumed. The collision must in fact have been an ordinary one ; critical cases will not do. In assuming a large area of contact we assume that the two spheres did not collide like elastic solid balls, but as fluid masses. This appears reasonable. The confinement of the vorticity introduced by frictional forces to a narrow region on both sides of the original layer of contact is a necessary result of hydrodynamical principles, and does not depend on the conservative forces operating, such as gravity. These might determine the motion of the masses as wholes ; the star, for instance, might depart in several pieces instead of one. But the present questions depend on the vortical motion in the layer of shearing, and appear in their general features to be independent of the type of motion away from the boundary, apart from the adopted velocity of the masses relative to each other.

In calculating the angular momentum of the ejected matter we

treated the interface as plane and the velocities within the shearing layer as all parallel to the original interface. Actually the interface would probably be curved. But the motion in the layer would be mainly parallel to the actual interface, and the calculation of the angular momentum of a small tube across the layer about its centre of mass could be found on this principle. If the tube had no linear momentum relative to a point, its angular momentum about that point would be the same. This appears to be an approximation to the actual situation. The mean velocity in the layer would be a mean of the velocities of the sun and the star, and so would the velocity of the centre of mass of the layer as a whole. The calculation of the angular momentum is therefore probably correct.

The theory of actual material collision between the sun and a star appears to involve nothing obviously unplausible, and it provides directly quantitative explanations of the total mass of the planets and their rates of rotation, and of the rate of rotation of the sun. It therefore seems to merit serious attention.

On the Recurrence of Magnetic Storms. By W. M. H. Greaves, M.A., and H. W. Newton. (Plate 9.)

(*Communicated by the Astronomer Royal.*)

1. In 1904 E. W. Maunder,* from a detailed study of the Greenwich magnetic records, showed that magnetic storms tend to recur at intervals of one synodic solar rotation (about 27 days). Maunder accordingly advanced the hypothesis that magnetic storms are due to the impingement on the earth of an agency proceeding in beams from limited portions of the sun's surface situated in the sunspot zones. A hypothesis of this nature had been previously suggested in 1896 by K. Birkeland,† who attributed storms to the action of beams of electrons proceeding from the sun.‡

The reality of the recurrence tendency was subsequently confirmed by C. Chree from a statistical discussion of magnetic character figures. More recently a further discussion on the same lines has been given by C. Chree and J. M. Stagg.§

In two previous papers ‖ we have given a discussion of the Greenwich solar and magnetic records for the period 1874–1927. For that purpose two catalogues of magnetic storms were prepared, comprising

(1) The greater magnetic storms, for which the ranges were equal

* *M.N.*, **65**, 2–34, 1904; see also *M.N.*, **65**, 538–559, 1905; **76**, 63–68, 1915.

† *Archives des Sciences phys. et nat. Geneva*, June 1896.

‡ The same suggestion was made in 1899 by A. Harvey (*Trans. Canadian Inst.*, vi, 345, 1898–99), who had noticed that in some cases storms followed each other at an interval of one solar rotation. See also *Observatory*, **28**, 100, 1905, for an account of some early suggestions of the indication of the solar rotation period by magnetic disturbances.

§ *Phil. Trans.*, A, **227**, 21–62, 1927. ‖ *M.N.*, **88**, 556, 1928; **89**, 84, 1928.

to or greater than 60′ in Declination or 300γ in Horizontal Force (or North Force for the years 1916 to 1925) or Vertical Force—60 storms.

(2) The lesser magnetic storms, for which the range in D was between 30′ and 60′, or that in H (or N) or V between 150γ and 300γ. This second list included a few storms which exceeded by a small amount the limit for the greater storms in one element only, and that by virtue of a single swing, such storms not having been included in the first list—343 storms.

These catalogues, which also contain relevant sunspot data, have been printed in the *Greenwich Photoheliographic Results* for 1927. The present communication aims at discussing these 403 storms from the standpoint of recurrence phenomena.

2. For the purpose of this discussion we introduce the conception of a "storm-day," by which is meant any calendar date on the whole or part of which one of the storms in either catalogue was taking place. For example, a storm was catalogued as 1891 September 27-29. September 27, 28, and 29 are accordingly all "storm-days."

The 403 storms were divided into five groups according to their intensities. The first group comprised the 60 "great" storms in the first catalogue. For the remaining storms, the mean of the ranges in the three components, D, H (or N), and V, was used as a criterion of intensity, the D range being converted into force for this purpose. These ranges are in fact given for each storm in the printed catalogues. The limiting ranges of the groups were selected so as to give approximately the same number of storms in each group.

Each of these five groups of storms was analysed as follows. For each storm the calendar date on which it commenced was called day 0; the following day was called day 1, and so on. For each group we have tabulated the percentage number of cases on which day n was a "storm day," and this analysis has been carried out from $n = 0$ to $n = 95$.

3. The results up to $n = 70$ are given in the diagram. The percentage numbers for the first few days after day 0 have, however, not been plotted so as to save space. These early percentage numbers are very large owing to the fact that the storms generally extend over two or more days.

An inspection of the diagram does not reveal any tendency of the "storm days" to be unusually frequent near $n = 27$ for the two groups of storms of greatest range. For the three remaining groups with mean range less than 180γ there is, however, a decided peak beginning near $n = 27$. Furthermore, in the two smallest groups a peak commencing near $n = 54$ makes its appearance. No significant peaks have been found near $n = 81$ for any of these groups, and this part of the diagram has not been reproduced.*

* The analysis of Chree and Stagg (*loc. cit.*, p. 34) shows up an 81-day peak, and also one at $n = 108$. These peaks are much smaller than those at $n = 27$ and $n = 54$, and would, in the present analysis, be masked by accidental variations. It should be noticed that the results of Chree and Stagg refer in the main to disturbances of smaller range than those discussed in the present paper (see remarks, *M.N.*, **89**, 87, 1928).

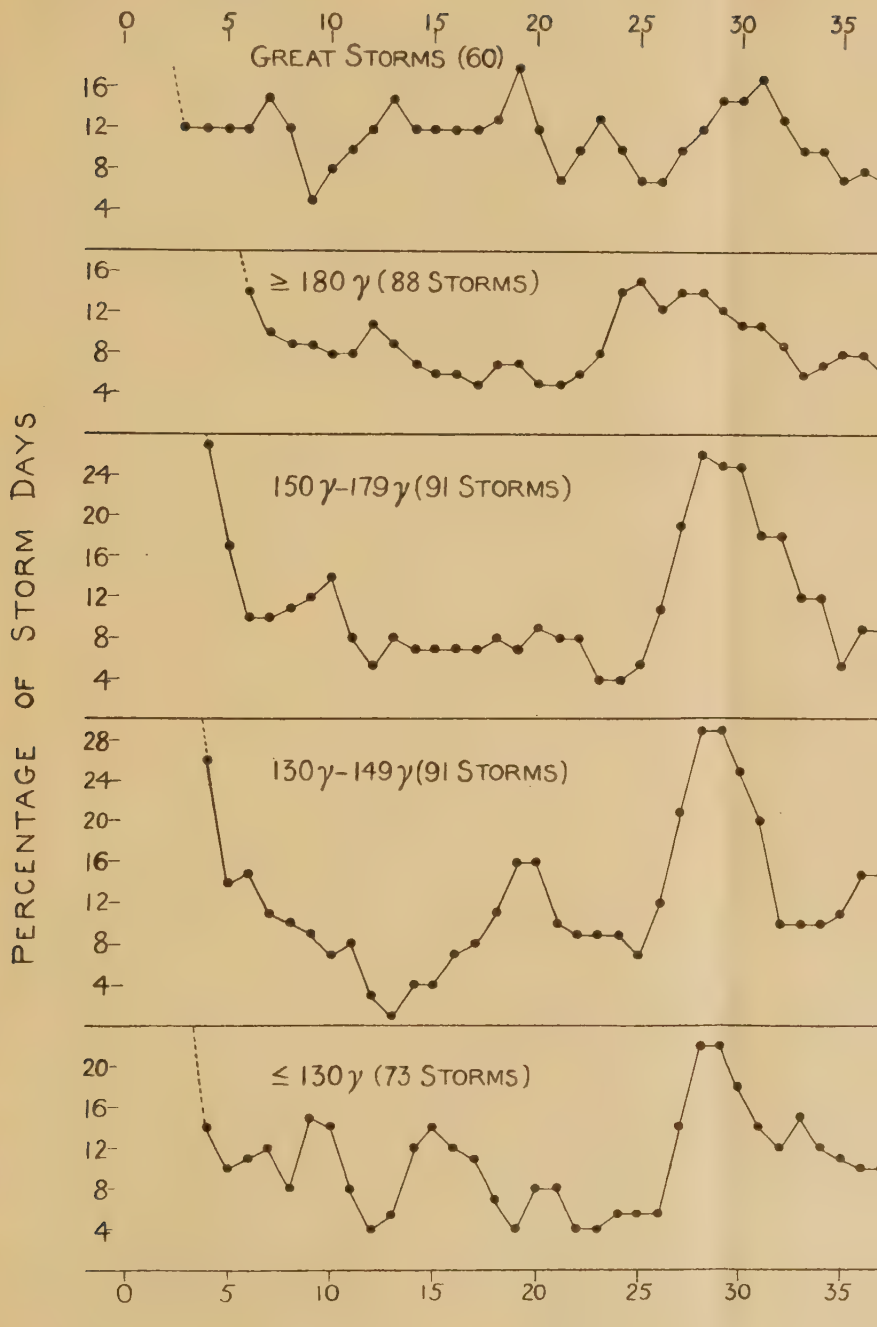
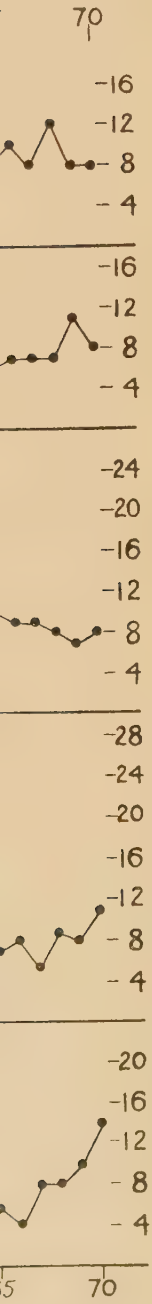

GREAT STORMS (60)
≥ 180 γ (88 STORMS)
150 γ–179 γ (91 STORMS)
130 γ–149 γ (91 STORMS)
≤ 130 γ (73 STORMS)
PERCENTAGE OF STORM DAYS
DAYS FROM COMME

PLATE 9.



It would thus appear that the recurrence characteristic is mainly a property of the storms of smaller range. The recurrence tendency has been used to ascribe magnetic storms to the action of radial streams proceeding from limited areas of the sun. The present investigation in no way contradicts this conclusion, but suggests an additional hypothesis, namely, that for the more intense storms the solar disturbance from which the stream is being ejected is in general short-lived, and does not survive a solar rotation in an appreciable number of cases.

We have tried the experiment of dividing the fourth group, for which the 27-day and 54-day peaks seem to be most marked, into two sub-groups. The peaks in question again appear in each sub-group, but the curves contain accidental variations of the same order of magnitude, and it is clear that nothing is to be gained by further subdivision.

4. It seems desirable to check the above conclusions by analysing the material from another point of view. For this purpose we have followed Maunder in picking out 27-day sequences of storms. The following table gives all the sequences containing at least three storms that can be formed from the storms in the catalogues, so that the interval between the catalogued times of commencement of two consecutive storms in a sequence lies between $25^{d}\cdot0$ and $29^{d}\cdot0$.

TABLE I.

27-Day Sequences ($25^{d}\cdot0$–$29^{d}\cdot0$).

Storm (Commencement).	Mean Range in γ.	Storm (Commencement).	Mean Range in γ.
I. 1883 Feb. 1·2	146	V. 1894 Apr. 17·5	198
Feb. 27·6	145	May 13·5	139
Mar. 26·8	147	June 8·9	213
Apr. 24·6	186	VI. 1894 Sept. 18·6	156
May 20·7	152	Oct. 16·1	117
II. 1886 Sept. 7·6	144	Nov. 13·6	G
Oct. 6·5	169	VII. 1895 Oct. 12·5	171
Nov. 2·7	144	Nov. 9·5	186
Nov. 29·6	124	Dec. 7·3	127
III. 1889 Sept. 8·8	116	1896 Jan. 2·9	142
Oct. 5·7	110	Jan. 29·8	136
Nov. 1·2	157	VIII. 1896 Oct. 8·5	185
Nov. 26·6	161	Nov. 5·8	155
IV. 1891 Sept. 27·8	139	Dec. 3·7	168
Oct. 23·8	132	IX. 1905 Jan. 5·6	119
Nov. 19·9	150	Feb. 3·1	195
		Mar. 2·6	152

TABLE I.—*continued.*

Storm (Commencement).	Mean Range in γ.	Storm (Commencement).	Mean Range in γ.
X. 1905 June 9·6	87	XVI. 1916 Mar. 29·2	135
July 5·9	135	Apr. 25·2	164
Aug. 2·0	112	May 21·0	150
Aug. 28·6	110	XVII. 1918 Oct. 15·5	194
XI. 1907 Jan. 14·8	148	Nov. 10·5	141
Feb. 9·6	G	Dec. 7·8	190
Mar. 10·2	120	1919 Jan. 3·8	161
XII. 1908 July 15·6	125	Jan. 31·4	119
Aug. 12·6	125	Feb. 27·8	146
Sept. 9·5	124	Mar. 27·5	136
XIII. 1908 Oct. 12·5	137	XVIII. 1920 Apr. 15·0	148
Nov. 7·8	152	May 13·0	115
Dec. 4·5	135	June 9·6	135
1909 Jan. 1·5	162	XIX. 1922 Sept. 7·3	133
Jan. 29·5	210	Oct. 4·8	163
XIV. 1909 Aug. 28·7	124	Oct. 30·8	136
Sept. 25·5	G	XX. 1926 Mar. 9·5	188
Oct. 23·0	167	Apr. 5·8	109
XV. 1910 Dec. 28·5	115	May 3·9	157
1911 Jan. 24·5	139	June 1·5	> 210
Feb. 21·5	163		
Mar. 20·0	134		

In three cases (denoted by the letter G) the mean range is not given, the storm in question being one of the 60 "great" storms for which, owing to frequent loss of trace, only lower limits for the ranges could in general be obtained. There are 75 storms involved in these 20 sequences, and it is rather striking that only 3 of these are "great" storms, in spite of the fact that in the catalogues from which the sequences were selected the proportion of "great" to all storms is 1 to 6·7.

TABLE II.

	Sequence Storms.	All Storms.	Ratio.
< 130 γ	18	73	0·25
130–149 γ	24	91	0·26
150–179 γ	19	91	0·21
> 180 γ	11	88	0·12
"Great storms"	3	60	0·05

Table II. gives for each of the five groups the total number of storms in that group, and also the number of such storms which appear in the sequences given in Table I. It will be seen that there is a progressive increase in the proportion of sequence storms as we pass from the " great " storms to the groups of smaller range, a result which is entirely in accord with the conclusion arrived at in paragraph 3.

5. So far we have been discussing the tendency or otherwise for one storm in our catalogues to be followed by another catalogued storm. Now these catalogues have excluded disturbances of intensities less than a certain limit, and there remains the possibility of the catalogued storms being followed 27 days later by subsidiary disturbance. In particular it is of interest to see whether there is any tendency for the larger magnetic storms, for which we have been unable to find a recurrence tendency, to be followed by subsidiary disturbance one synodic rotation later.

The Greenwich magnetic traces usually extend from noon to noon. The ranges of the Declination traces have been measured for the 25th, 26th, 27th, 28th, and 29th traces after the trace on which each storm commenced, those cases in which one of the catalogued storms was taking place on any of these traces being excluded from these measurements.

Table III.

Limits of Storm.	Number.	Means of Declination Ranges of Traces.				
		25.	26.	27.	28.	29.
		′	′	′	′	′
$<130\gamma$	53	10·1	11·5	12·1	12·8	12·2
$130–149\gamma$	59	11·2	11·2	12·0	13·8	13·3
$150–179\gamma$	61	11·5	11·9	12·4	12·3	13·1
$>180\gamma$	64	11·5	12·1	14·0	13·8	12·4
Great storms	47	12·8	13·4	13·2	12·2	12·3

In Table III. these ranges have been meaned for the various groups. There seems to be a slight tendency for the ranges to increase from the 25th trace onwards, but bearing in mind that the average range in Declination on an ordinary day is about $12\frac{1}{2}'$, it becomes apparent that there is no very appreciable tendency for storms, which are not followed a solar rotation later by a further disturbance of storm range, to be followed by any subsidiary disturbance.

6. At first sight it is rather paradoxical that the storms of larger range should not exhibit the recurrence tendency, for it is precisely these storms which seem to show a connection with sunspots larger than the average, and in view of the marked tendency of the larger spots to persist for more than one rotation, it might have been expected that the largest storms would show the recurrence tendency best. Table IV. is reproduced from a previous paper * (*q.v.* for details), and shows the

* *M.N.*, **89**, 87, 1928. Some small computational errors in the original table have now been corrected.

average areas of the largest sunspots on the sun's disc at the times of the storms both for the central portion of the disc and for the zones near the limbs.

TABLE IV.

Average Areas of Largest Sunspots at Time of Commencement of Magnetic Storms.

Limits of Storm.	Spots near Centre of Disc 53° E. to 53° W.		Spots near Sun's Limbs 53° to 79° E. and W.		Number of Storms.
	Average Area.	Relative Values.	Average Area.	Relative Values.	
"Quiet days"	244	1·0	167	1·0	264 (Quiet days.)
$<120\gamma$	260	1·1	181	1·1	42
$120–150\gamma$	282	1·2	165	1·0	122
$150–180\gamma$	318	1·3	211	1·3	91
$180–210\gamma$	390	1·6	302	1·8	52
$>210\gamma$	462	1·9	189	1·1	36
Great storms	691	2·8	251	1·5	43
Very great storms	1116	4·6	181	1·1	17

(Areas are corrected for foreshortening.)

On comparing the data in this table with the diagram, there is seen to be a marked antithesis between the tendency to recurrence and the tendency for the storms to be associated with spots. The paradox, however, is more apparent than real, as the occurrence of storms during spotless periods * shows that the spots themselves are not the actual seat of origin of the streams responsible for the storms. All that can be said is that there is a tendency for the solar disturbances responsible for the larger storms to be accompanied by spots larger than the average, and once it has been granted that the disturbances are not identical with the spots, there seems to be no serious difficulty in assuming that the larger disturbances may subside comparatively quickly although an associated spot may persist. In one respect the two sets of data are supplementary, as they both seem to indicate that the storms are connected with a disturbance on a limited area of the sun's surface, the recurrence tendency suggesting this for the smaller storms, whereas the correlation with sunspot data points to the same conclusion in the case of the larger storms.

* *Loc. cit.*, p. 89.

Royal Observatory, Greenwich :
1929 *April* 29.

On the Dwarf Nature of Double Stars in General.

By E. A. Kreiken.

Introduction.—If at a certain stage a star splits up into a double star it is evident that the two components which result in this way must have relatively small masses, *i.e.* small absolute luminosities. I therefore believe that in a general way the components of a double star will belong to the main series of dwarfs.

With those physically connected pairs of which we have a more or less accurate knowledge as to the individual values of mass and magnitude, viz. the Algol or eclipsing variables,* the visual binaries with known orbits,† and the spectroscopic binaries,‡ this actually appears to be the case.

With the double stars in general we have to use statistical methods. In the present paper the writer has compiled all observational evidence which he has obtained in several papers as to the dwarf nature of the double stars in general. Before inspecting these observational data we must first consider a few theoretical questions.

Let the absolute luminosity curve of the double stars be

$$\log \phi(\mathrm{M}) = p + q\mathrm{M} - r\mathrm{M}^2 \quad . \quad . \quad . \quad (1)$$

The density function is

$$\log \mathrm{D}(\rho) = h + k\rho - l\rho^2 \quad . \quad . \quad . \quad (2)$$

where

$$\rho = 5 \log r.$$

The distribution of the absolute distances between the two components of a double star at the unit of distance is taken equal to

$$\log \psi(\mathrm{L}) = u + v\mathrm{L} - w\mathrm{L}^2 \quad . \quad . \quad . \quad (3)$$

where $\mathrm{L} = \log d$ (d being the angular distance between the two components expressed in seconds of arc).

Then the apparent distribution curve of the double stars with one and the same distance r will be

$$\psi(\mathrm{L}) = 10^{u+(\mathrm{L}+0\cdot2\rho)v-w(\mathrm{L}+0\cdot2\rho)^2}.$$

The number of double stars with apparent magnitude m and with an angular distance $\mathrm{L} = \log d$ will be equal to

$$\mathrm{D}(m, d) = \text{const.} \int_{-\infty}^{+\infty} 10^{u+(\mathrm{L}+0\cdot2\rho)v-(\mathrm{L}+0\cdot2\rho)^2w+p+q(m-\rho)-r(m-\rho)^2+h+k\rho-l\rho^2+0\cdot6\rho} d\rho \quad (4)$$

The solution of this integral has been given in the *B.A.N.*, **154**, 249, where we found

$$\mathrm{D}(m, d) = 10^{\mathrm{A}+\mathrm{B}m-\mathrm{C}m^2+\mathrm{D}\mathrm{L}-\mathrm{E}\mathrm{L}^2-\mathrm{F}m\mathrm{L}} \quad . \quad . \quad (5)$$

* G. Shajn, *M.N.*, **88**, 213, 1928.
† A. Brill, *Veröff. Berlin-Babelsberg.*
‡ E. A. Kreiken, *M.N.*, **89**, 589, 1929.

in which

$$\left.\begin{aligned} \mathrm{F} &= \frac{+\,0{\cdot}04rw}{(l+0{\cdot}04w+r)}. \\ \mathrm{E} &= w-\frac{0{\cdot}04w^2}{(l+0{\cdot}04w+r)}. \\ \mathrm{D} &= v-\frac{0{\cdot}2w(-q+0{\cdot}2v+k+0{\cdot}6)}{(l+0{\cdot}04w+r)}. \\ \mathrm{C} &= r-\frac{r^2}{(l+0{\cdot}04w+r)}. \\ \mathrm{B} &= q+\frac{r(-q+0{\cdot}2v+k+0{\cdot}6)}{(l+0{\cdot}04w+r)}. \end{aligned}\right\} \quad . \quad . \quad (6)$$

From the formulæ (5) and (6) it appears that the distribution of the double stars over the different magnitudes is not completely determined by the absolute luminosity curve (1) and the density function (2), but is also systematically dependent on the distribution curve of the absolute distances (3).

As a matter of fact it must be evident that the increase with magnitude of the numbers of double stars, for instance with $d < 5''$, will be a combined effect of two causes: (*a*) the increase of the double star numbers of one and the same absolute distance D, with increasing magnitude; (*b*) the increase of the total number of double stars with D smaller than a certain limit, with increasing value of that limit.

Though not explicitly stated, it will need no further explanation that our formulæ (5) and (6) only hold good when for all magnitudes equal intervals of magnitude difference, Δm, have been observed. As a rule the numbers of stars with large Δm are but incompletely observed, and if this fact is not sufficiently taken into account we obtain erroneous results.

One additional conclusion may be drawn from our formulæ. Suppose that the absolute luminosity function and the density function of the double stars and those of the stars in general are the same, then owing to the function (3) the apparent luminosity curve of the double stars (5) must still be different from that of the stars in general. So different apparent luminosity curves must be obtained in the various galactic latitudes for both groups of stars, even though all other distribution functions are equal. If, therefore, the galactic condensation of the double stars is compared with that of the stars in general, we cannot expect to obtain the same values. Therefore no conclusions as to the absolute luminosity of the double stars can be drawn from their galactic condensation.

I have discussed these theoretical questions in some detail in order to point out that it is very dangerous to draw far-reaching conclusions from incomplete material. We will now discuss the observational data.

The Apparent Luminosity Curves of the Double Stars.—In the *B.A.N.*, **151**, **152**, and **154**, I gave the distribution of the double stars down to magnitude 12·0 over different magnitudes, magnitude differences, and

angular distances. These results were based on the double-star catalogues which J. Scheiner,* Father J. Stein,† and Dr. H. Groot ‡ have drawn up from the Potsdam, Vatican, and Greenwich Astrographic Catalogues respectively. After applying corrections for the numbers of optical double stars we obtained the apparent luminosity curves as given in Table I. In Table II. these apparent luminosity curves are compared with the luminosity curves of the stars in general. For the latter the necessary data were obtained from the tables by Seares, Van

Table I.

Apparent Luminosity Curves of Double Stars in Different Galactic Latitudes. For each group of double stars Δm *was taken equal to* 2·4. *The magnitudes are on the int. phot. scale. Surface,* 1000 *square degrees.*

log ΣD(*p*)*m*.

d.	<5″.			5″–10″.			10″–15″.			0″–15″.		
m.	0°–20°.	20°–40°.	40°–60°.	0°–20°.	20°–40°.	40°–60°.	0°–20°.	20°–40°.	40°–60°.	0°–20°.	20°–40°.	40°–60°.
	9·362	..	..	9·146	9·653	..	9·477	..	..	9·826	9·653	..
7·0												
	9·663	..	..	9·763	9·875	9·556	0·146	0·093	..	0·387	0·301	9·556
8·0												
	0·519	9·968	..	0·380	0·130	9·954	0·544	0·518	9·398	0·962	0·748	0·061
9·0												
	0·959	0·568	0·400	1·025	0·695	0·491	0·881	0·748	0·230	1·436	1·154	0·863
10·0												
	1·408	1·185	1·121	1·427	1·161	0·996	1·226	0·978	0·785	1·840	1·593	1·467
11·0												
	1·918	1·573	1·514	1·954	1·581	1·458	1·659	1·332	1·314	2·339	1·987	1·914
12·0												

Table II.

Differences between log D(*m*) *and* log N(*m*) (log D(*m*) − log N(*m*) + 3) *in Different Latitudes. The values* log N(*m*) *were taken from the work of Seares, Van Rhijn, Joyner, and Richmond.*

	β=0°–20°.				β=20°–40°.				β=40°–60°.			
d.	<5″.	5″–10″.	10″–15″.	0″–15″.	<5″.	5″–10″.	10″–15″.	0″–15″.	<5″.	5″–10″.	10″–15″.	0″–15″.
m.	−·116	−·332	−·001	+·348	..	+·403	..	+·403	..	..	..	..
7·0												
	−·257	−·157	+·226	+·467	..	+·181	+·399	+·607	..	+·011	..	+·011
8·0												
	+·181	−·022	+·186	+·604	−·160	+·002	+·390	+·620	..	−·024	−·580	+·083
9·0												
	+·168	+·234	+·090	+·645	+·017	+·144	+·197	+·603	+·005	+·096	−·165	+·468
0·0												
	+·181	+·210	+·009	+·623	+·221	+·197	+·014	+·629	+·326	+·201	−·010	+·672
1·0												
	+·284	+·320	+·024	+·703	+·209	+·217	−·032	+·623	+·336	+·280	+·136	+·736
2·0												

* *Publ. Potsdam*, Nr. 59, 1908.

† *Catalogo Astrografico, Sesione Vaticana*, Appendice III.

‡ Only a short account has been published up till now (*M.N.*, **88**, 51–56, 1927). The Astronomer Royal and Dr. Groot kindly gave me results to use in advance of publication.

Rhijn, Joyner, and Richmond.* The values given in Table II. are equal to

$$\log D_p(m) - \log N_1(m) + .3,$$

where $D_p(m)$ and $N_1(m)$ are the numbers of double stars and of the stars at large respectively.

It appears that the relative frequency of the double stars increases with increasing magnitude. For the double stars with $d < 5''$ this partly at least is due to the increasing absolute value of D corresponding to the apparent angular distance d with increasing magnitude (see Introduction).

But as we also find an increase of the star numbers with $d = 5''$–$10''$ we may conclude that this also is partly due to the increase, with increasing magnitude, of the percentages of the stars which are double stars. This is what would be expected if the double stars were dwarfs. With increasing magnitude of the stars in general, the fractions of these stars which are dwarfs are increasing. In his brilliant monograph on double stars, Aitken † finds that the relative frequency of the double stars decreases with increasing magnitude. I think that this must be due to the fact that the visual catalogues are incomplete for the faint double stars with large magnitude difference.

The Distribution of the Binary Stars by Spectral Class.—Aitken (*loc. cit.*) has given the frequency of the different spectral classes among the binary stars down to magnitude 9·0 (*Bonner Durchmusterung*).

Without taking into account the various spectral subdivisions, his results are the following :—

Table III.

Spectral Type.	O.	B.	A.	F.	G.	K.	M.
Observed numbers	5	325	1219	1153	758	466	13

The present author has investigated the frequency of the double stars which are contained in the Southern Double-star Catalogue. A detailed account of these results will shortly be published in the *Bulletin of the Astronomical Institutes of the Netherlands.* Table IV. contains a short summary of them. We consider only the spectral types B, A, F, G, and K. The spectral type B is very frequent both with the double stars and the stars in general in the lower latitudes, while in the higher latitudes only a few B stars are found. The spectral type A is very frequent with both groups of stars. The difference in the distribution curves, however, sets in with the later types. With the ordinary stars the F and G type stars are of intermediate frequency, but the K stars are again very abundant.

With the double stars, however, we observe a continuous decrease of the star numbers with the F, G, and K stars. Double stars of spectral type K are very rare. This also is apparent from Aitken's numbers in Table III.

* *Astroph. Journ.*, **62**, 320 and following, 1925.
† R. Aitken, *The Binary Stars*, p. 262, table vii.

TABLE IV.

Distribution of Double Stars by Spectral Type (numbers per 1000 square degrees).

Sp. Type \ Latitude.	Double Stars. 0°–20°.	20°–40°.	40°–90°.	Stars in general. 0°–20°.	20°–40°.	40°–90°.
B	19·44	5·35	0·97	324	79	22
A	23·76	12·39	9·72	502	200	132
F	13·32	10·45	10·93	205	204	168
G	6·30	5·59	6·32	169	162	151
K	4·32	3·65	1·70	380	240	240
			Percentages.			
B	29	14	3	21	9	3
A	35	33	33	32	23	18
F	20	28	37	13	23	24
G	9	15	21	11	18	21
K	7	10	6	23	27	34

It is not difficult to explain this peculiar difference. The bulk of the stars contained in the Henry Draper Catalogue are giants. F and G giants do but seldom occur, which explains the low numbers of these stars in the distribution curves of the stars in general. On the other hand, the double stars are mainly dwarfs, and therefore the usual sequence of spectral types B, A, F, G, and K indicates stars of ever-decreasing absolute luminosity, *i.e.* ever-increasing magnitude. The volume of space which is covered by our observations therefore decreases with the later types. So the relative frequency of the latter must decrease.

The Frequency of Angular Distances d.—From the observational material we may obtain one more proof of the double stars being dwarfs.

We concluded that the usual sequence of spectral types B, A, F, G, and K indicates stars of ever-decreasing luminosity. Consequently the mean absolute luminosity of the stars of one and the same apparent magnitude, m, must systematically be dependent on spectral type, viz. this luminosity is largest for the B stars and smallest for the K ones. In other words, the mean distance of the stars of one and the same magnitude must decrease with the later spectral types. Owing to this the mean angular distance between the components of the double stars will increase with the later types.

In Table V. I have given for different latitudes the distribution of the double stars of different spectral types over the angular distances d. Only double stars with apparent magnitude 7·0 8·0 are taken into account. The result is as was anticipated, that with the later types the relative frequency of the stars with $d < 1''$ decreases, while the numbers with $d = 5''$–$15''$ increase. Owing to the small numbers the values in the higher latitudes show considerable fluctuations, but nevertheless the evidence is conclusive. It should be remarked that it is not possible to derive from the observed distribution curve $\psi(L)$ the

actual one, as the observed numbers are largely influenced by systematic errors. If, for instance, a bright star has a faint companion, it will be difficult to detect the latter if the apparent angular distance is small. On the other hand, very large distances are but incompletely observed. I cannot think, however, of any possible reason why these two effects should vary with spectral type. Therefore they cannot have any possible influence on the preceding conclusions.

TABLE V.

Distribution of the Double Stars of Various Spectral Types by Angular Distance d. (*Magnitude* 7·0–8·0.)

Lat.	0°–20°			20°–40°			40°–90°		
d. / Sp. Type.	<1″.	1″–5″.	5″–15″.	<1″.	1″–5″.	5″–15″.	<1″.	1″–5″.	5″–15″.
B	38	40	22	..	..	..	..	..	..
A	36	40	24	35	38	27	65	24	11
F	33	44	23	49	27	24	27	47	26
G	24	40	36	22	44	34	37	32	31
K	25	38	37	30	10	60	..	..	..

The Galactic Concentration of the Double Stars.—The galactic concentration of the double stars has been frequently studied (*e.g.* Aitken, *loc. cit.*, and Kreiken, *B.A.N.*, **154**), and it has been usually found that the galactic concentration of the double stars is somewhat larger than that of the stars in general. We need not enter into details. In the introduction it has been pointed out that it is very dangerous to draw from this any conclusions as to the absolute luminosity of the stars. It may, however, be useful to remark that, if the galactic concentration of the double stars with d smaller than a certain limit is compared with that of ordinary stars of the same absolute magnitude, the result must be that a higher concentration is obtained for the double stars. When we consider the double stars with $d < 5''$ we only take into account those binary systems whose actual distance between the two components is smaller than const. $\times 5r$ (r being the distance of the system). In other words, the fraction of the total numbers of double stars which are considered increases with distance. In the higher latitudes the density falls off much more quickly with increasing distance than in the lower ones, and the result is an excess of concentration of double stars.

Frequency of Physical Pairs.—It has been frequently remarked that the spectroscopic binaries are most frequent among the earlier types.

At present I do not think there is any evidence that the relative frequency of either visual or spectroscopic binaries is different for the different spectral classes, nor is there at present any reason to suppose that the ratio of the numbers of visual double stars to the numbers of spectroscopic binaries will vary with spectral type. From the foregoing it appears that, if we want to compare the relative frequency of the different spectral types with the ordinary stars and with the double

stars, only the dwarfs of the former should be taken into account. If, on the other hand, we compare the relative frequencies of visual and spectroscopic binaries, the same limiting magnitude should be adopted for both groups of stars. It is apparent that for both groups of binary stars the relative frequencies are systematically dependent on this limiting magnitude. With the fainter magnitudes the fractions of the stars which are dwarfs will increase, and this increase will vary with spectral type.

From the foregoing considerations it does not follow that there will occur no single binary among the giant stars, but we conclude that they must be very infrequent.

Jeans * and ten Bruggencate † have pointed out that double stars may develop from the variables of the giant series. If, therefore, giant double stars occur, it would be expected that their spectral types are earlier than those of the variables. In this respect the diagrams derived by P. Doig,‡ F. C. Leonard,§ and G. Shajn ‖ are very remarkable. A possible explanation has, however, been given by H. Siedentopf.¶

Summary.—1. Evidence has been given in various ways that the double stars are mainly dwarfs.

(*a*) The frequency of double stars rapidly increases with magnitude.
(*b*) The distribution of spectral types with double stars is quite different from the corresponding distribution with the stars in general. This difference is easily explained if it is accepted that the double stars are dwarfs.
(*c*) With the later spectral types the mean angular separation between the two components of the double stars increases.

2. From the galactic concentration of the double stars no conclusions can be drawn as to their absolute magnitudes.

3. We have no evidence that the relative frequency of either the visual or spectroscopic binaries should be different for the different spectral types. Nor is it permissible to conclude that the distribution of the visual binaries over different spectral types is different from the distribution of the spectroscopic binaries. The preference of the spectroscopic binaries for the earlier spectral types is due to the selection of very bright stars. The "dwarf" spectroscopic binaries of the later types are but incompletely observed.

* *M.N.*, **85**, 797, 1925.
† *Naturwissenschaften*, **14**, 905, 1926.
‡ *M.N.*, **82**, 372, 1921.
§ *L.O.B.*, **10**, 169, 1923.
‖ *A.N.*, 5404, 49, 1925.
¶ *Veröff. Univ. Sternw. Gött.*, Heft 3, 1928.

Bosscha Observatory,
Lembang (Java):
1929 *April.*

Atlas Stellarum Variabilium, Series VII. By S. A. Mitchell.

The appearance from the Vatican Observatory of *Atlas Stellarum Variabilium*, Series VII, is a notable event in the history of variable stars. This series, started by Father Hagen while at Georgetown Observatory in the United States, was interrupted by his transfer to the Vatican Observatory, and was still further postponed pending the completion of the Vatican section of the Astrographic Catalogue.

As is well known, the purpose of the Hagen charts has been to derive the magnitudes of a sequence of stars to be used as comparison stars for long-period variables. In Series VII there are forty-one charts, involving fifty-four variables. All of the forty-one regions are found in *Harvard Annals*, **57**, and form an extension of the Harvard work to fainter stars. The completed task consists of four separate and distinct parts :

(1) The making of grade estimates of all stars of the different sequences by Father Hagen.

(2) The measurement by means of the photometer by Graff at the Bergedorf Observatory of a series of photometric standards.

(3) The plotting of the grade estimates against the magnitudes photometrically determined. Thus were derived magnitudes for all of the comparison stars. This part of the work was done by Father Esch of Valkenburg, Holland.

(4) The measurement of the co-ordinates of the comparison stars from photographs. This also was done by Father Esch.

For many years the Leander McCormick Observatory has been engaged in a similar project by means of its 66-cm. visual refractor, and in consequence has extended the Harvard sequences to stars fainter than the fifteenth magnitude (when necessary) in no less than 188 variable star regions. Unfortunately, on account of press of other work, the results have not yet been published. However, beginning with 1929 January 1, the American Association of Variable Star Observers have adopted all of the McCormick magnitudes in all of the regions investigated, and in consequence a new series of charts for the use of its members has been prepared and distributed.

The methods followed at the McCormick Observatory have been identical with those outlined above for Series VII. Briefly the plan was as follows :—

(1) Alden and Mitchell, working together with a wedge photometer, derived a series of standards by comparisons with stars photometrically measured at Harvard.

(2) Alden and Mitchell, working separately, observed the grade estimates of each of the stars of the different series.

(3) Grade estimates were plotted against photometric magnitudes. Part of this work was done by Alden alone, the balance by Mitchell, the two sharing about equally in the observations and the reductions.

The method of procedure of the McCormick Observatory is described in greater detail in *Monthly Notices*, **86**, 356, 1926. It was there shown

that in the region of Nova Persei the McCormick magnitudes of the comparison stars have the small probable error of ±0·04 magnitudes, while "the McCormick scale is 1 per cent. less extended than the photo-visual scale of Mount Wilson, this latter scale being in substantial accord with the international standard."

Of the forty-one regions of Series VII, twenty-seven have been finished at the McCormick Observatory. As the results are absolutely and completely independent, comparisons will be of great value. In its work the McCormick Observatory desired merely to secure a sequence of stars of accurately determined values to be used as comparison stars, and hence there are fewer McCormick stars than there are in Series VII. In carrying out the programme, the McCormick observers have had the advantage of a telescope with larger aperture than is ordinarily used in systematic work on variable stars. The evident result has been that the faintest stars of the sequences have been observed more readily.

Altogether in the twenty-seven regions there are a total of 384 stars in common, or an average of fourteen stars for each of the regions observed. The differences in the sense McCormick minus Series VII were taken for each of the 384 common stars, with the results given in the following tables.

In the first table the stars of all twenty-seven regions are grouped together according to magnitudes. In the first column are the McCormick magnitudes, in the second are the average differences taken without regard to sign, in the third the average systematic differences in the sense McCormick minus Series VII. The plus sign indicates that the McCormick value is the fainter. The fourth column shows the number of stars in common.

Table I.

Magnitude.	Average Difference.	Systematic Difference.	Number.
> 8·5	·147	+·035	17
8·6– 9·0	·132	−·037	19
9·1– 9·5	·207	+·100	28
9·6–10·0	·135	−·038	37
10·1–10·5	·141	+·020	41
10·6–11·0	·183	−·011	36
11·1–11·5	·172	−·117	37
11·6–12·0	·148	−·007	47
12·1–12·5	·209	+·027	34
12·6–13·0	·203	−·023	36
13·1–13·5	·287	−·061	31
13·6–14·0	·314	−·029	15
<14·1	·266	−·266	6
Means	·186	−·016	384

In the second table the stars are arranged according to regions. A word of comment might be added. The average magnitude for the

faintest star in each region measured by the photometer at Harvard is 12·24. In two of the regions, TY Cygni, 192928, and RS Aquilæ, 1953*08*, the systematic differences—McCormick minus Series VII—are rather large, being due to differences in zero-point. In the first of the two regions the faintest star measured by photometer at Harvard is only 11·16, while in the second region the Harvard measures in the years 1905 and 1906 are very discordant.

TABLE II.

Designation.	Variable.	Average Difference.	Systematic Difference.	Number.
054974	V Camelop	·244	−·094	16
1302*12*	RV Virginis	·164	−·128	11
140113	Z Bootis	·150	−·110	10
165631	RV Herculis	·170	+·030	10
170627	RT Herculis	·167	−·044	18
175519	RY Herculis	·183	−·012	17
175654	V Draconis	·147	·000	15
180565	W Draconis	·100	−·038	13
180666	X Draconis			
183225	RZ Herculis	·173	−·133	15
185634	Z Lyræ	·146	+·080	13
185737	RT Lyræ	·155	+·064	11
190529	V Lyræ	·194	−·065	17
190941	RU Lyræ	·143	−·029	13
190967	U Draconis	·192	−·038	13
190925	S Lyræ	·261	+·117	18
190933a	RS Lyræ	·212	−·050	16
192928	TY Cygni	·314	+·186	14
193311	RT Aquilæ	·254	+·192	13
193509	RV Aquilæ	·217	−·133	12
194348	TU Cygni	·167	−·078	9
194604	X Aquilæ	·164	−·022	14
1952*02*	RR Aquilæ	·146	−·023	13
1953*08*	RS Aquilæ	·236	+·236	14
200812	RU Aquilæ	·144	−·094	16
202954	ST Cygni	·161	−·039	18
211614	X Pegasi	·183	−·047	19
233956	Z Cassiopeiæ	·181	+·031	16
	Means	·186	−·016	384

There are many chances for making large systematic and large accidental differences between McCormick magnitudes and those of Series VII. In calibrating the Bergedorf and McCormick photometers it is easily possible for the two scales to be different. In the comparisons actually carried out for each region, different Harvard stars were observed as standards by the two photometers, with the possibility of a difference in zero-point creeping into the magnitudes in addition to a systematic difference in scale. The estimates of grades of the sequences in each region involves constancy in observing qualities which can come only as the result of long training. Naturally, the weakest parts of the sequences are at the faintest stars, and for two reasons : (1) It is necessary to extrapolate beyond the point where photometer observations are possible, and (2) the stars are nearly at the limit of visibility of the telescope.

In spite of all the different possibilities of errors, Tables I and II show that the magnitudes from Series VII and the McCormick values agree in a surprising manner. For all twenty-seven of the regions observed in common there is a negligible systematic difference amounting to ·016 magnitude. For all differences of the 384 stars taken without regard to sign, the average value is $\pm$·186 magnitude. If we assume that the magnitudes of Series VII and McCormick have equal weight, then either system of magnitudes is reliable within 0·13 magnitude. If a few of the differences in zero-point were adjusted (which could readily be done), and a few large differences at the faint end of the scale discarded, then the magnitudes of both Series VII and McCormick can be relied upon to within about one-tenth of a magnitude.

The comparisons in Table I seem to show that the scale of Series VII is more extended than the McCormick scale by the slight amount of 1·3 per cent. Hence, both scales are in excellent agreement with the photovisual scale of Mount Wilson, or with the international standard scale.

In addition to the 384 stars included in the above discussion there are about half a dozen stars where differences of a whole magnitude are found between the three authorities, Series VII, Harvard, or McCormick. Since each of the three have made use of both photometer and grade estimates, it is comparatively easy to decide in these discordances whether the large differences are due to variability, or are to be ascribed to misidentification or to poor observations.

Summary.

1. The results published in Series VII of *Atlas Stellarum Variabilium*, and also those from the McCormick Observatory, are each an extension to fainter stars of the magnitude system of comparison stars in *Harvard Annals*, **57**.

2. The magnitudes of 384 stars observed in common in twenty-seven different regions of long-period variables have been determined by similar methods.

3. The scales of Series VII and McCormick are in excellent agreement with the international standard scale.

4. The systematic difference between the two sets of magnitudes is negligible.

5. The accidental differences of either series of magnitudes are little in excess of one-tenth of a magnitude.

6. The writer wishes to pay high tribute to the excellent series of splendid researches on variable stars inaugurated by Father Hagen, the " dean " of variable star observers of to-day.

Leander McCormick Observatory,
University of Virginia:
1929 *March* 7.